Hello AP Physics C!

I look forward to seeing you all this fall. AP Physics C: Mechanics will tackle the fascinating problems which you considered in AP Physics 1 with more advanced mathematical techniques. For some of you, this will be the first time using Calculus, for others, it will be an opportunity to apply Calculus in a new and exciting way.

In order to hit the ground running, I would like you all to take some time to consider the idea of vectors. You surely have seen many vectors in your time in AP Physics 1, but you may not have always treated them in the most rigorous mathematical way. To make sure we can take full advantage of our college-level textbook, I would like you to take the time to try the following questions which I copied from our textbook and attached after this message.

1.31
1.46
1.52
1.54
1.66
1.70
1.82
1.95

Only 8 problems! However, to make sure that you are comfortable with things like i, j, k notation or taking dot and cross products, I have attached a nice summary of everything you want to know about vectors from the University College London. Please take the time to read it and grapple with these 8 problems.

Do not be overly discouraged if you find this challenging, vector algebra is far from the most important aspect of our class; however, this course is supposed to be about going deeper mathematically into the ideas we learned in Physics 1.

I will be collecting this homework on the first day for a grade. If you need help, feel free to email me at mhammond@eustace.org

Sincerely,
Dr. Hammond

1.29 How much would it cost to paper the entire United States (including Alaska and Hawaii) with dollar bills? What would be the cost to each person in the United States?

Section 1.7 Vectors and Vector Addition

1.30 Hearing rattles from a snake, you make two rapid displacements of magnitude 1.8 m and 2.4 m. In sketches (roughly to scale), show how your two displacements might add up to give a resultant of magnitude a) 4.2 m; b) 0.6 m; c) 3.0 m.

1.31 A postal employee drives a delivery truck along the route shown in Fig. 1.26. Determine the magnitude and direction of the resultant displacement by drawing a scale diagram. (See also Exercise 1.38 for a different approach to this same problem.)

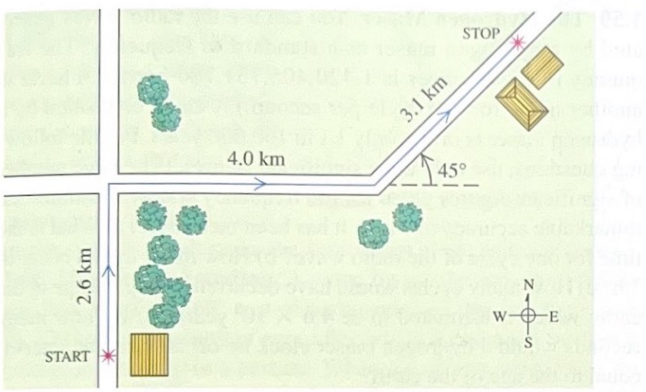


Figure 1.26 Exercises 1.31 and 1.38.

1.32 For the vectors $\vec{A}$ and $\vec{B}$ in Fig. 1.27, use a scale drawing to find the magnitude and direction of a) the vector sum $\vec{A} + \vec{B}$; b) the vector difference $\vec{A} - \vec{B}$. From your answers to parts (a) and (b), find the magnitude and direction of c) $-\vec{A} - \vec{B}$; d) $\vec{B} - \vec{A}$. (See also Exercise 1.39 for a different approach to this problem.)

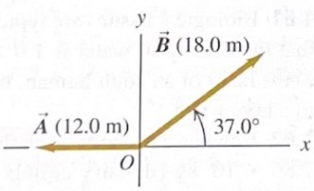


Figure 1.27 Exercises 1.32, 1.39, 1.44, and 1.54.

1.33 A spelunker is surveying a cave. She follows a passage 180 m straight west, then 210 m in a direction 45° east of south, and then 280 m at 30° east of north. After a fourth unmeasured displacement she finds herself back where she started. Use a scale drawing to determine the magnitude and direction of the fourth displacement. (See also Problem 1.69 for a different approach to this problem.)

Section 1.8 Components of Vectors

1.34 Use a scale drawing to find the x- and y-components of the following vectors. For each vector the numbers given are i) the magnitude of the vector and ii) the angle, measured in the sense from the $+x$-axis toward the $+y$-axis, that it makes with the $+x$-axis. Find a) magnitude 9.30 m, angle 60.0°; b) magnitude 22.0 km, angle 135°; c) magnitude 6.35 cm, angle 307°.

1.35 Compute the x- and y-components of the vectors $\vec{A}$, $\vec{B}$, and $\vec{C}$ in Fig. 1.28.

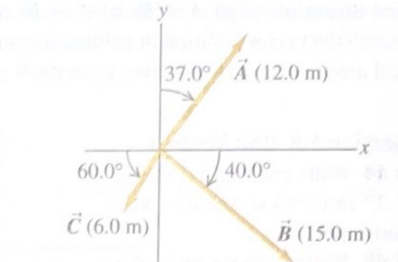


Figure 1.28 Exercises 1.35, 1.45, and 1.50, and Problem 1.68.

1.36 Let the angle θ be the angle that the vector $\vec{A}$ makes with the $+x$-axis, measured counterclockwise from that axis. Find the angle θ for a vector that has the following components: a) $A_x = 2.00$ m, $A_y = -1.00$ m; b) $A_x = 2.00$ m, $A_y = 1.00$ m; c) $A_x = -2.00$ m, $A_y = 1.00$ m; d) $A_x = -2.00$ m, $A_y = -1.00$ m.

1.37 A rocket fires two engines simultaneously. One produces a thrust of 725 N directly forward while the other gives a 513 N thrust at 32.4° above the forward direction. Find the magnitude and direction (relative to the forward direction) of the resultant force which these engines exert on the rocket.

1.38 A postal employee drives a delivery truck over the route shown in Fig. 1.26. Use the method of components to determine the magnitude and direction of her resultant displacement. In a vector addition diagram (roughly to scale), show that the resultant displacement found from your diagram is in qualitative agreement with the result you obtained using the method of components.

1.39 For the vectors $\vec{A}$ and $\vec{B}$ in Fig. 1.27, use the method of components to find the magnitude and direction of a) the vector sum $\vec{A} + \vec{B}$; b) the vector sum $\vec{B} + \vec{A}$; c) the vector difference $\vec{A} - \vec{B}$; d) the vector difference $\vec{B} - \vec{A}$.

1.40 Find the magnitude and direction of the vector represented by the following pairs of components: a) $A_x = -8.60$ cm, $A_y = 5.20$ cm; b) $A_x = -9.70$ m, $A_y = -2.45$ m; c) $A_x = 7.75$ km, $A_y = -2.70$ km.

1.41 A disoriented physics professor drives 3.25 km north, then 4.75 km west, and then 1.50 km south. Find the magnitude and direction of the resultant displacement, using the method of components. In a vector addition diagram (roughly to scale), show that the resultant displacement found from your diagram is in qualitative agreement with the result you obtained using the method of components.

1.42 Vector $\vec{A}$ has components $A_x = 1.30$ cm, $A_y = 2.25$ cm; vector $\vec{B}$ has components $B_x = 4.10$ cm, $B_y = -3.75$ cm. Find a) the components of the vector sum $\vec{A} + \vec{B}$; b) the magnitude and direction of $\vec{A} + \vec{B}$; c) the components of the vector difference $\vec{B} - \vec{A}$; d) the magnitude and direction of $\vec{B} - \vec{A}$.

1.43 Vector $\vec{A}$ is 2.80 cm long and is 60.0° above the x-axis in the first quadrant. Vector $\vec{B}$ is 1.90 cm long and is 60.0° below the x-axis in the fourth quadrant (Fig. 1.29). Find the magnitude

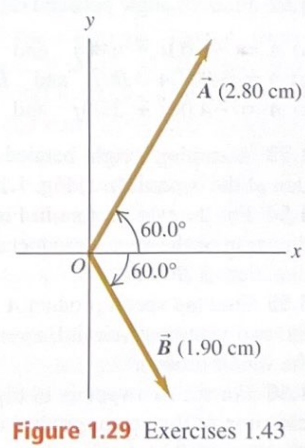


Figure 1.29 Exercises 1.43 and 1.56.

and direction of a) $\vec{A} + \vec{B}$; b) $\vec{A} - \vec{B}$; c) $\vec{B} - \vec{A}$. In each case, sketch the vector addition or subtraction and show that your numerical answers are in qualitative agreement with your sketch.

Section 1.9 Unit Vectors

1.44 Write each vector in Fig. 1.27 in terms of the unit vectors $\hat{\imath}$ and $\hat{\jmath}$.

1.45 Write each vector in Fig. 1.28 in terms of the unit vectors $\hat{\imath}$ and $\hat{\jmath}$.

1.46 a) Write each vector in Fig. 1.30 in terms of the unit vectors $\hat{\imath}$ and $\hat{\jmath}$. b) Use unit vectors to express the vector $\vec{C}$, where $\vec{C} = 3.00\vec{A} - 4.00\vec{B}$. c) Find the magnitude and direction of $\vec{C}$.

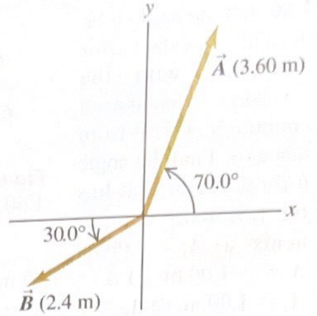


Figure 1.30 Exercise 1.46 and Problem 1.82.

1.47 Given two vectors $\vec{A} = 4.00\hat{\imath} + 3.00\hat{\jmath}$ and $\vec{B} = 5.00\hat{\imath} - 2.00\hat{\jmath}$, a) find the magnitude of each vector; b) write an expression for the vector difference $\vec{A} - \vec{B}$ using unit vectors; c) find the magnitude and direction of the vector difference $\vec{A} - \vec{B}$. d) In a vector diagram show $\vec{A}$, $\vec{B}$, and $\vec{A} - \vec{B}$, and also show that your diagram agrees qualitatively with your answer in part (c).

1.48 a) Is the vector $(\hat{\imath} + \hat{\jmath} + \hat{k})$ a unit vector? Justify your answer. b) Can a unit vector have any components with magnitude greater than unity? Can it have any negative components? In each case justify your answer. c) If $\vec{A} = a(3.0\hat{\imath} + 4.0\hat{\jmath})$, where a is a constant, determine the value of a that makes $\vec{A}$ a unit vector.

1.49 a) Use vector components to prove that two vectors commute for both addition and the scalar product. b) Use vector components to prove that two vectors anticommute for the vector product. That is, prove that $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$.

Section 1.10 Products of Vectors

1.50 For the vectors $\vec{A}$, $\vec{B}$, and $\vec{C}$ in Fig. 1.28, find the scalar products a) $\vec{A} \cdot \vec{B}$; b) $\vec{B} \cdot \vec{C}$; c) $\vec{A} \cdot \vec{C}$.

1.51 a) Find the scalar product of the two vectors $\vec{A}$ and $\vec{B}$ given in Exercise 1.47. b) Find the angle between these two vectors.

1.52 Find the angle between each of the following pairs of vectors:

a) $\vec{A} = -2.00\hat{\imath} + 6.00\hat{\jmath}$ and $\vec{B} = 2.00\hat{\imath} - 3.00\hat{\jmath}$
b) $\vec{A} = 3.00\hat{\imath} + 5.00\hat{\jmath}$ and $\vec{B} = 10.00\hat{\imath} + 6.00\hat{\jmath}$
c) $\vec{A} = -4.00\hat{\imath} + 2.00\hat{\jmath}$ and $\vec{B} = 7.00\hat{\imath} + 14.00\hat{\jmath}$

1.53 Assuming a right-handed coordinate system, find the direction of the $+z$-axis in a) Fig. 1.15a; b) Fig. 1.15b.

1.54 For the two vectors in Fig. 1.27, a) find the magnitude and direction of the vector product $\vec{A} \times \vec{B}$; b) find the magnitude and direction of $\vec{B} \times \vec{A}$.

1.55 Find the vector product $\vec{A} \times \vec{B}$ (expressed in unit vectors) of the two vectors given in Exercise 1.47. What is the magnitude of the vector product?

1.56 For the two vectors in Fig. 1.29, a) find the magnitude and direction of the vector product $\vec{A} \times \vec{B}$; b) find the magnitude and direction of $\vec{B} \times \vec{A}$.

Problems

1.57 An acre, a unit of land measurement still in wide use, has a length of one furlong ($\frac{1}{8}$ mi) and a width one-tenth of its length. a) How many acres are in a square mile? b) How many square feet are in an acre? See Appendix E. c) An acre-foot is the volume of water that would cover one acre of flat land to a depth of one foot. How many gallons are in an acre-foot?

1.58 An estate on the California coast was offered for sale for \$4,950,000. The total area of the estate was 102 acres (see Problem 1.57). a) Considering the price of the estate to be proportional to its area, what was the cost of one square meter of the estate? b) What would be the price of a portion of the estate the size of a postage stamp ($\frac{7}{8}$ in. by 1.0 in.)?

1.59 The Hydrogen Maser. You can use the radio waves generated by a hydrogen maser as a standard of frequency. The frequency of these waves is 1,420,405,751.786 hertz. (A hertz is another name for one cycle per second.) A clock controlled by a hydrogen maser is off by only 1 s in 100,000 years. For the following questions, use only three significant figures. (The large number of significant figures given for the frequency simply illustrates the remarkable accuracy to which it has been measured.) a) What is the time for one cycle of the radio wave? b) How many cycles occur in 1 h? c) How many cycles would have occurred during the age of the earth, which is estimated to be 4.6×10^9 years? d) By how many seconds would a hydrogen maser clock be off after a time interval equal to the age of the earth?

1.60 Estimate the number of atoms in your body. (*Hint:* Based on what you know about biology and chemistry, what are the most common types of atom in your body? What is the mass of each type of atom? Appendix D gives the atomic masses for different elements, measured in atomic mass units; you can find the value of an atomic mass unit, or 1 u, in Appendix F.)

1.61 Biological tissues are typically made up of 98% water. Given that the density of water is 1.0×10^3 kg/m^3, estimate the mass of a) the heart of an adult human; b) a cell with a diameter of 0.5 μm; c) a honey bee.

1.62 Iron has a property such that a 1.00 m^3 volume has a mass of 7.86×10^3 kg (density equals 7.86×10^3 kg/m^3). You want to manufacture iron into cubes and spheres. Find a) the length of the side of a cube of iron that has a mass of 200 g; b) the radius of a solid sphere of iron that has a mass of 200 g.

1.63 a) Estimate the number of dentists in your city. You will need to consider the number of people in your city, how often they need to go to the dentist, how often they actually go, how many hours a typical dental procedure (filling, root canal, and so on) takes, and how many hours a dentist works in a week. b) Using your local telephone directory, check to see whether your estimate was roughly correct.

1.64 Physicists, mathematicians, and others often deal with large numbers. The number 10^{100} has been given the whimsical name *googol* by mathematicians. Let us compare some large numbers in physics with the googol. (*Note:* This problem requires numerical values that you can find in the appendices of the book, with which you should become familiar.) a) Approximately how many atoms make up our planet? For simplicity, assume the average atomic mass of the atoms to be 14 g/mol. Avogadro's number gives the number of atoms in a mole. b) Approximately how many neutrons are in a neutron star? Neutron stars are composed almost entirely of neutrons and have approximately twice the mass of the sun. c) In the leading theory of the origin of the universe, the entire universe that we can now observe occupied, at a very early time, a sphere whose radius was approximately equal to the present distance of the earth to the sun. At that time the universe had a density (mass divided by volume) of 10^{15} g/cm^3. Assuming that $\frac{1}{3}$ of the particles were protons, $\frac{1}{3}$ of the particles were neutrons, and the remaining $\frac{1}{3}$ were electrons, how many particles then made up the universe?

1.65 Three horizontal ropes pull on a large stone stuck in the ground, producing the vector forces $\vec{A}$, $\vec{B}$ and $\vec{C}$ shown in Fig. 1.31. Find the magnitude and direction of a fourth force on the stone that will make the vector sum of the four forces zero.

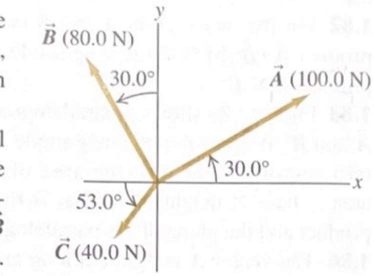


Figure 1.31 Problem 1.65.

1.66 Emergency Landing. A plane leaves the airport in Galisto and flies 170 km at 68° east of north and then changes direction to fly 230 km at 48° south of east, after which it makes an immediate emergency landing in a pasture. When the airport sends out a rescue crew, in which direction and how far should this crew fly to go directly to this plane?

1.67 You are to program a robotic arm on an assembly line to move in the xy-plane. Its first displacement is $\vec{A}$; its second displacement is $\vec{B}$, of magnitude 6.40 cm and direction 63.0° measured in the sense from the $+x$-axis toward the $-y$-axis. The resultant $\vec{C} = \vec{A} + \vec{B}$ of the two displacements should also have a magnitude of 6.40 cm, but a direction 22.0° measured in the sense from the $+x$-axis toward the $+y$-axis. a) Draw the vector addition diagram for these vectors, roughly to scale. b) Find the components of $\vec{A}$. c) Find the magnitude and direction of $\vec{A}$.

1.68 a) Find the magnitude and direction of the vector $\vec{R}$ that is the sum of the three vectors $\vec{A}$, $\vec{B}$, and $\vec{C}$ in Fig. 1.28. In a diagram, show how $\vec{R}$ is formed from the three vectors in Fig. 1.28. b) Find the magnitude and direction of the vector $\vec{S} = \vec{C} - \vec{A} - \vec{B}$. In a diagram, show how $\vec{S}$ is formed from the three vectors in Fig. 1.28.

1.69 As noted in Exercise 1.33, a spelunker is surveying a cave. She follows a passage 180 m straight west, then 210 m in a direction 45° east of south, then 280 m at 30° east of north. After a fourth unmeasured displacement she finds herself back where she started. Use the method of components to determine the magnitude and direction of the fourth displacement. Draw the vector addition diagram and show that it is in qualitative agreement with your numerical solution.

1.70 A sailor in a small sailboat encounters shifting winds. She sails 2.00 km east, then 3.50 km southeast, and then an additional distance in an unknown direction. Her final position is 5.80 km directly east of the starting point (Fig. 1.32). Find the magnitude and direction of the third leg of the journey. Draw the vector addition diagram and show that it is in qualitative agreement with your numerical solution.

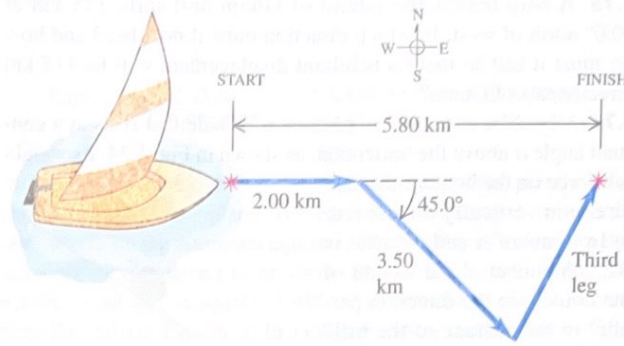


Figure 1.32 Problem 1.70.

1.71 A cross-country skier skis 2.80 km in the direction 45.0° west of south, then 7.40 km in the direction 30.0° north of east, and finally 3.30 km in the direction 22.0° south of west. a) Show these displacements in a diagram. b) How far is the skier from the starting point?

1.72 On a training flight, a student pilot flies from Lincoln, Nebraska to Clarinda, Iowa, then to St. Joseph, Missouri, and then to Manhattan, Kansas (Fig. 1.33). The directions are shown relative to north: 0° is north, 90° is east, 180° is south, and 270° is west. Use the method of components to find a) the distance she has to fly from Manhattan to get back to Lincoln; b) the direction (relative to north) she must fly to get there. Illustrate your solution with a vector diagram.

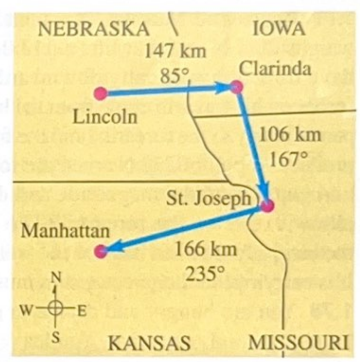


Figure 1.33 Problem 1.72.

1.73 A graphic artist is creating a new logo for her company's Web site. In the graphics program she is using, each pixel in an image file has coordinates (x, y), where the origin (0, 0) is at the upper-left corner of the image, the $+x$-axis points to the right, and the $+y$-axis points down. Distances are measured in pixels. a) The artist draws a line from the pixel location (10, 20) to the location (210, 200). She wishes to draw a second line that starts at (10, 20), is 250 pixels long, and is at angle of 30° measured clockwise from the first line. At which pixel location should this second line end? Give your answer to the nearest pixel. b) The artist now draws an arrow that connects the lower-right end of the first line to the lower right end of the second line. Find the length and direction of this arrow. Draw a diagram showing all three lines.

1.74 Getting Back. An explorer in the dense jungles of equatorial Africa leaves his hut. He takes 40 steps northeast, then 80 steps 60° north of west, then 50 steps due south. Assume his steps all have equal length. a) Sketch, roughly to scale, the three vectors and their resultant. b) Save him from becoming hopelessly lost in the jungle by giving him the displacement, calculated using the method of components, that will return him to his hut.

1.75 A ship leaves the island of Guam and sails 285 km at 40.0° north of west. In which direction must it now head and how far must it sail so that its resultant displacement will be 115 km directly east of Guam?

1.76 A boulder of weight w rests on a hillside that rises at a constant angle α above the horizontal, as shown in Fig. 1.34. Its weight is a force on the boulder that has direction vertically downward. a) In terms of α and w, what is the component of the weight of the boulder in the direction parallel to the surface of the hill? b) What is the component of the weight in the direction perpendicular to the surface of the hill? c) An air conditioner unit is fastened to a roof that slopes upward at an angle of 35.0°. In order that the unit not slide down the roof, the component of the unit's weight parallel to the roof cannot exceed 550 N. What is the maximum allowed weight of the unit?

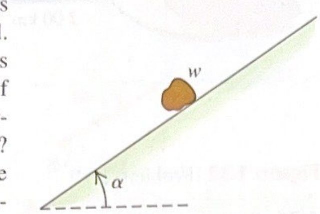


Figure 1.34 Problem 1.76.

1.77 Bones and Muscles. A patient in therapy has a forearm that weighs 20.5 N and that lifts a 112.0-N weight. These two forces have direction vertically downward. The only other significant forces on his forearm come from the biceps muscle (which acts perpendicularly to the forearm) and the force at the elbow. If the biceps produces a pull of 232 N when the forearm is raised 43° above the horizontal, find the magnitude and direction of the force that the elbow exerts on the forearm. (The sum of the elbow force and the biceps force must balance the weight of the arm and the weight it is carrying, so their vector sum must be 132.5 N, upward.)

1.78 You are hungry and decide to go to your favorite neighborhood fast-food restaurant. You leave your apartment and take the elevator 10 flights down (each flight is 3.0 m) and then go 15 m south to the apartment exit. You then proceed 0.2 km east, turn north, and go 0.1 km to the entrance of the restaurant. a) Determine the displacement from your apartment to the restaurant. Use unit-vector notation for your answer, being sure to make clear your choice of coordinates. b) How far did you travel along the path you took from your apartment to the restaurant and what is the magnitude of the displacement you calculated in part (a)?

1.79 You are canoeing on a lake. Starting at your camp on the shore, you travel 240 m in the direction 32° south of east to reach a store to purchase supplies. You know the distance because you have located both your camp and the store on a map of the lake. On the return trip you travel distance B in the direction 48° north of west, distance C in the direction 62° south of west, and then you are back at your camp. You measure the directions of travel with your compass, but you don't know the distances. Since you are curious to know the total distance you rowed, use vector methods to calculate the distances B and C.

1.80 You are camping with two friends, Joe and Karl. Since all three of you like your privacy, you don't pitch your tents close together. Joe's tent is 21.0 m from yours, in the direction 23.0° south of east. Karl's tent is 32.0 m from yours, in the direction 37.0° north of east. What is the distance between Karl's tent and Joe's tent?

1.81 Vectors $\vec{A}$ and $\vec{B}$ are drawn from a common point. Vector $\vec{A}$ has magnitude A and angle θ_A measured in the sense from the $+x$-axis to the $+y$-axis. The corresponding quantities for vector $\vec{B}$ are B and θ_B. Then $\vec{A} = A\cos\theta_A\hat{\imath} + A\sin\theta_A\hat{\jmath}$, $\vec{B} = B\cos\theta_B\hat{\imath} + B\sin\theta_B\hat{\jmath}$, and $\phi = |\theta_B - \theta_A|$ is the angle between $\vec{A}$ and $\vec{B}$. a) Derive Eq. (1.18) from Eq. (1.21). b) Derive Eq. (1.22) from Eq. (1.27).

1.82 For the two vectors $\vec{A}$ and $\vec{B}$ in Fig. 1.30, a) find the scalar product $\vec{A}\cdot\vec{B}$; b) find the magnitude and direction of the vector product $\vec{A}\times\vec{B}$.

1.83 Figure 1.8c shows a parallelogram based on the two vectors $\vec{A}$ and $\vec{B}$. a) Show that the magnitude of the cross product of these two vectors is equal to the area of the parallelogram. (*Hint:* area = base × height.) b) What is the angle between the cross product and the plane of the parallelogram?

1.84 The vector $\vec{A}$ is 3.50 cm long and is directed into this page. Vector $\vec{B}$ points from the lower-right corner of this page to the upper-left corner of this page. Define an appropriate right-handed coordinate system and find the three components of the vector product $\vec{A}\times\vec{B}$, measured in cm^2. In a diagram, show your coordinate system and the vectors $\vec{A}$, $\vec{B}$, and $\vec{A}\times\vec{B}$.

1.85 Given two vectors $\vec{A} = -2.00\hat{\imath} + 3.00\hat{\jmath} + 4.00\hat{k}$ and $\vec{B} = 3.00\hat{\imath} + 1.00\hat{\jmath} - 3.00\hat{k}$, do the following. a) Find the magnitude of each vector. b) Write an expression for the vector difference $\vec{A} - \vec{B}$, using unit vectors. c) Find the magnitude of the vector difference $\vec{A} - \vec{B}$. Is this the same as the magnitude of $\vec{B} - \vec{A}$? Explain.

1.86 Bond Angle in Methane. In the methane molecule, CH_4, each hydrogen atom is at a corner of a regular tetrahedron with the carbon atom at the center. In coordinates where one of the C—H bonds is in the direction of $\hat{\imath} + \hat{\jmath} + \hat{k}$, an adjacent C—H bond is in the $\hat{\imath} - \hat{\jmath} - \hat{k}$ direction. Calculate the angle between these two bonds.

1.87 The two vectors $\vec{A}$ and $\vec{B}$ are drawn from a common point, and $\vec{C} = \vec{A} + \vec{B}$. a) Show that if $C^2 = A^2 + B^2$, the angle between the vectors $\vec{A}$ and $\vec{B}$ is 90°. b) Show that if $C^2 < A^2 + B^2$, the angle between the vectors $\vec{A}$ and $\vec{B}$ is greater than 90°. c) Show that if $C^2 > A^2 + B^2$, the angle between the vectors $\vec{A}$ and $\vec{B}$ is between 0° and 90°.

1.88 When two vectors $\vec{A}$ and $\vec{B}$ are drawn from a common point, the angle between them is ϕ. a) Using vector techniques, show that the magnitude of their vector sum is given by

$$\sqrt{A^2 + B^2 + 2AB\cos\phi}$$

b) If $\vec{A}$ and $\vec{B}$ have the same magnitude, for which value of ϕ will their vector sum have the same magnitude as $\vec{A}$ or $\vec{B}$? c) Derive a result analogous to that in part (a) for the magnitude of the vector difference $\vec{A} - \vec{B}$. d) If $\vec{A}$ and $\vec{B}$ have the same magnitude, for what value of ϕ will $\vec{A} - \vec{B}$ have this same magnitude?

1.89 A cube is placed so that one corner is at the origin and three edges are along the x-, y-, and z-axes of a coordinate system (Fig. 1.35). Use vectors to compute a) the angle between the edge along the z-axis (line ab) and the diagonal from the origin to the opposite corner (line ad); b) the angle between line ac (the diagonal of a face) and line ad.

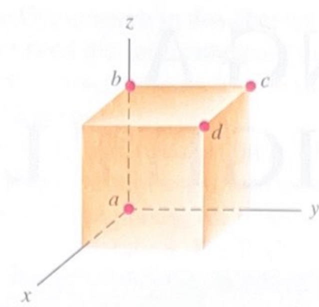


Figure 1.35 Problem 1.89.

1.90 Obtain a *unit vector* perpendicular to the two vectors given in Problem 1.85.

1.91 You are given vectors $\vec{A} = 5.0\hat{\imath} - 6.5\hat{\jmath}$ and $\vec{B} = -3.5\hat{\imath} + 7.0\hat{\jmath}$. A third vector $\vec{C}$ lies in the xy-plane. Vector $\vec{C}$ is perpendicular to vector $\vec{A}$, and the scalar product of $\vec{C}$ with $\vec{B}$ is 15.0. From this information, find the components of vector $\vec{C}$.

1.92 Two vectors $\vec{A}$ and $\vec{B}$ have magnitude $A = 3.00$ and $B = 3.00$. Their vector product is $\vec{A} \times \vec{B} = -5.00\hat{k} + 2.00\hat{\imath}$. What is the angle between $\vec{A}$ and $\vec{B}$?

1.93 Later in our study of physics we will encounter quantities represented by $(\vec{A} \times \vec{B}) \cdot \vec{C}$. a) Prove that for any three vectors $\vec{A}$, $\vec{B}$, and $\vec{C}$, $\vec{A} \cdot (\vec{B} \times \vec{C}) = (\vec{A} \times \vec{B}) \cdot \vec{C}$. b) Calculate $(\vec{A} \times \vec{B}) \cdot \vec{C}$ for the three vectors $\vec{A}$ with magnitude $A = 5.00$ and angle $\theta_A = 26.0°$ measured in the sense from the $+x$-axis toward the $+y$-axis, $\vec{B}$ with $B = 4.00$ and $\theta_B = 63.0°$, and $\vec{C}$ with magnitude 6.00 and in the $+z$-direction. Vectors $\vec{A}$ and $\vec{B}$ are in the xy-plane.

Challenge Problems

1.94 The length of a rectangle is given as $L \pm l$, and its width as $W \pm w$. a) Show that the uncertainty in its area A is $a = Lw + lW$. Assume that the uncertainties l and w are small; so that the product lw is very small and you can ignore it. b) Show that the fractional uncertainty in the area is equal to the sum of the fractional uncertainty in length and the fractional uncertainty in width. c) A rectangular solid has dimensions $L \pm l$, $W \pm w$, *and* $H \pm h$. Find the fractional uncertainty in the volume, and show that it equals the sum of the fractional uncertainties in the length, width, and height.

1.95 Completed Pass. At Enormous State University (ESU), the football team records its plays using vector displacements, with the origin taken to be the position of the ball before the play starts. In a certain pass play, the receiver starts at $+1.0\hat{\imath} - 5.0\hat{\jmath}$, where the units are yards, $\hat{\imath}$ is to the right, and $\hat{\jmath}$ is downfield. Subsequent displacements of the receiver are $+9.0\hat{\imath}$ (in motion before the snap), $+11.0\hat{\jmath}$ (breaks downfield), $-6.0\hat{\imath} + 4.0\hat{\jmath}$ (zigs), and $+12.0\hat{\imath} + 18.0\hat{\jmath}$ (zags). Meanwhile, the quarterback has dropped straight back to a position $-7.0\hat{\jmath}$. How far and in which direction must the quarterback throw the ball? (Like the coach, you will be well advised to diagram the situation before solving it numerically.)

1.96 Navigating in the Solar System. The *Mars Polar Lander* spacecraft was launched on January 3, 1999. On December 3, 1999, the day that *Mars Polar Lander* touched down on the Martian surface, the positions of the earth and Mars were given by these coordinates:

	x	y	z
Earth	0.3182 AU	0.9329 AU	0.0000 AU
Mars	1.3087 AU	−0.4423 AU	−0.0414 AU

In these coordinates, the sun is at the origin and the plane of the earth's orbit is the xy-plane. The earth passes through the $+x$-axis once a year on the autumnal equinox, the first day of autumn in the northern hemisphere (on or about September 22). One AU, or *astronomical unit*, is equal to 1.496×10^8 km, the average distance from the earth to the sun. a) In a diagram, show the positions of the sun, the earth, and Mars on December 3, 1999. b) Find the following distances in AU on December 3, 1999: i) from the sun to the earth; ii) from the sun to Mars; iii) from the earth to Mars. c) As seen from the earth, what was the angle between the direction to the sun and the direction to Mars on December 3, 1999? d) Explain whether Mars was visible from your location at midnight on December 3, 1999. (When it is midnight at your location, the sun is on the opposite side of the earth from you.)

1.97 Navigating in the Big Dipper. All the stars of the Big Dipper (part of the constellation Ursa Major) may appear to be the same distance from the earth, but in fact they are very far from each other. Figure 1.36 shows the distances from the earth to each of these stars. The distances are given in light years (ly), the distance that light travels in one year. One light year equals 9.461×10^{15} m. a) Alkaid and Merak are 25.6° apart in the earth's sky. In a diagram, show the relative positions of Alkaid, Merak, and our sun. Find the distance in light years from Alkaid to Merak. b) To an inhabitant of a planet orbiting Merak, how many degrees apart in the sky would Alkaid and our sun be?

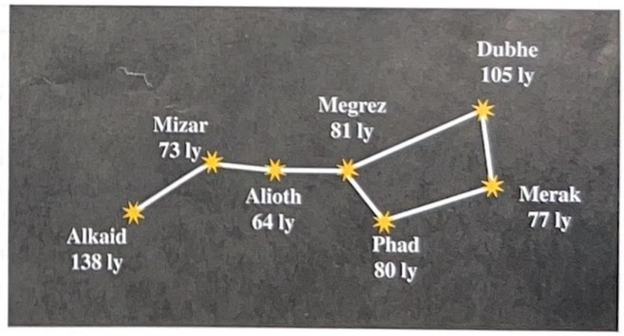


Figure 1.36 Challenge Problem 1.97.

1.98 The vector $\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$, called the *position vector*, points from the origin (0, 0, 0) to an arbitrary point in space with coordinates (x, y, z). Use what you know about vectors to prove the following: All points (x, y, z) that satisfy the equation $Ax + By + Cz = 0$, where A, B, and C are constants, lie in a plane that passes through the origin and that is perpendicular to the vector $A\hat{\imath} + B\hat{\jmath} + C\hat{k}$. Sketch this vector and the plane.

Chapter 6

Vectors

6.1 Introduction

Definition 6.1. *A* vector *is a quantity with both a* <u>*magnitude*</u> *(size) and* <u>*direction*</u>*.*

Many quantities in engineering applications can be described by vectors, e.g. force, velocity, magnetic field.

They can be represented by arrows, for example...

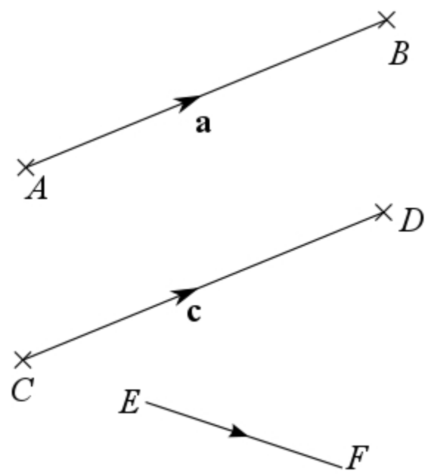


Figure 6.1: Some vectors.

Magnitude=Length of AB

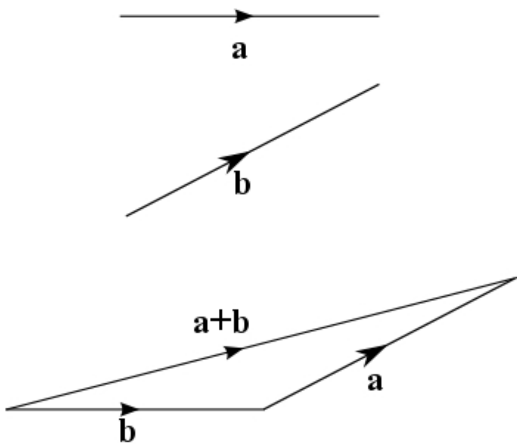
Direction is shown in the Figure 6.1.

We will write $\overrightarrow{AB}$ or $\mathbf{a}$ to represent the top vector in the figure.

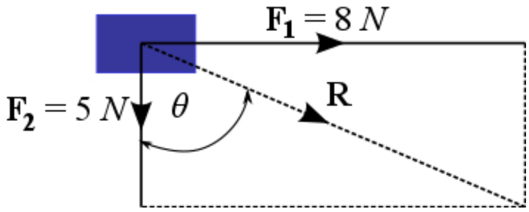
Two vectors are equal when they have both the same magnitude and direction. So $\overrightarrow{AB} = \overrightarrow{CD}$.

But $\overrightarrow{AB} \neq \overrightarrow{EF}$, since both the magnitude and direction are different.

The sum of two vectors **a** and **b** is found by adding the vectors "head to tail":



Example 6.1 (Forces on an object)**.** Consider the following forces acting on an object:



Forces add to give a net effect or resultant force.

$$\mathbf{R} = \mathbf{F_1} + \mathbf{F_2}$$

$$\text{Magnitude: } |\mathbf{R}| = \sqrt{8^2 + 5^2} \approx 9.4\text{N}.$$

$$\text{Direction: Use } \tan\theta = \frac{|\mathbf{F_1}|}{|\mathbf{F_2}|} = \frac{8}{5} = 1.6$$
$$\Rightarrow \theta = 58°.$$

You can multiply a vector **a** by a scalar (number) k. Then, as shown in Figure 6.2, if $k > 0$, $k\mathbf{a}$ is a vector in the same direction as **a**, and the magnitude is $k|\mathbf{a}|$... BUT if $k < 0$, $k\mathbf{a}$ is in the opposite direction!

Example 6.2. Two points A and B have position vectors (i.e. relative to a fixed origin O) **a** and **b** respectively. What is the position vector of a point on the line joining A and B, equidistant from A and B?

Well, the first thing we need is a sketch of the problem, like in Figure 6.3.

Next, note that $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$.

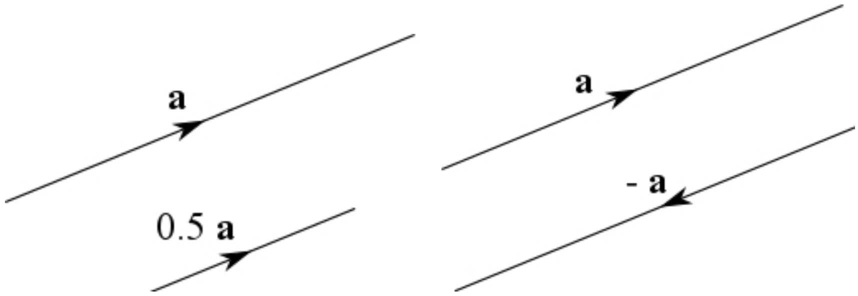


Figure 6.2: Two examples of scalar multiplication of the vector **a**.

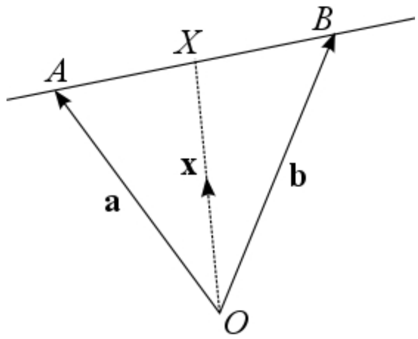


Figure 6.3: In this sketch, X is the midpoint of the line joining A and B

$$
\begin{aligned}
\mathbf{x} &= \mathbf{a} + \overrightarrow{AX} = \mathbf{a} + \frac{1}{2}\overrightarrow{AB} \\
&= \mathbf{a} + \frac{1}{2}(\mathbf{b} - \mathbf{a}) \\
&= \frac{1}{2}(\mathbf{a} + \mathbf{b}).
\end{aligned}
$$

Definition 6.2. *A* unit vector *is a vector with magnitude 1.*

Often represented using a hat symbol:

For any vector **a**,

$$
\hat{\mathbf{a}} = \frac{\mathbf{a}}{|\mathbf{a}|} \quad \text{is a unit vector since}
$$

$$
|\mathbf{a}| = \left| \frac{\mathbf{a}}{|\mathbf{a}|} \right| = \frac{|\mathbf{a}|}{|\mathbf{a}|} = 1.
$$

<u>Unit</u> vectors in the x, y, z idrections are denoted **i**, **j**, **k** respectively.

Then the position of a point P from the origin, with coordinates (x, y, z), is

$$
\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.
$$

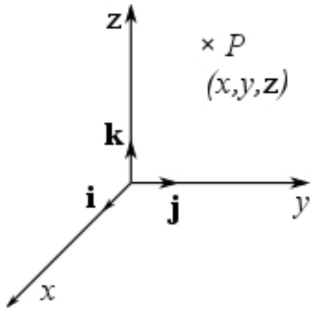


Figure 6.4: ijk

Example 6.3.

$$\mathbf{a} = 6\mathbf{i} - 3\mathbf{j} + \mathbf{k},$$
$$\mathbf{b} = 4\mathbf{i} + 2\mathbf{j}.$$

Then

$$\mathbf{a} + \mathbf{b} = 10\mathbf{i} - \mathbf{j} - \mathbf{k}$$
$$\mathbf{b} - \mathbf{a} = -2\mathbf{i} + 5\mathbf{j} - \mathbf{k}$$
$$3\mathbf{a} = 18\mathbf{i} - 9\mathbf{j} + 3\mathbf{k}.$$

For a position vector $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, the magnitude is

$$|\mathbf{r}| = \sqrt{x^2 + y^2 + z^2}.$$

Then for the previous example,

$$|\mathbf{a}| = \sqrt{6^2 + (-3)^2 + 1^2} = \sqrt{46},$$
$$|\mathbf{b}| = \sqrt{4^2 + 2^2 + 0^2} = 2\sqrt{5}.$$

So far we've seen how to add two vectors. Now we have a question...

Q: How can we multiply two vectors together?

I'm going to show you that there are in fact <u>two</u> ways to multiply vectors...

6.2 The Dot Product

Let us consider the origin of the dot product:

We take two vectors **a** and **b**:

We might be interested in the length of the component of **a** which is in the same direction as **b**.

Here $0 \leq \theta < \pi$ is the angle between **a** and **b**.

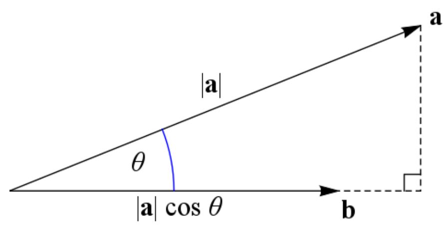


Figure 6.5: The two vectors **a** *and* **b**. *We see that the length of the component of* **a** *which is in the same direction as* **b** *is* $|\mathbf{a}|\cos\theta$.

Compare with the *dot product* formula:

$$\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$$

Looks almost like the length of the component of **a**, but is rescaled such that we have the symmetry:

$$\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a}$$

So the dot product also gives us a rescaling of the length of the component of **b** in the same direction as **a**. But we expected that in the first place, because of the above symmetry rule!

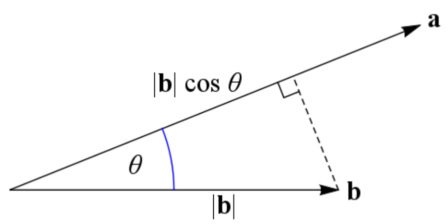


Figure 6.6: This time, we would like the length of the component of **b** *which is in the same direction as* **a**. *That length is* $|\mathbf{b}|\cos\theta$.

Note that

$$\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta \quad \Rightarrow \quad \cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|};$$

which is a useful method for calculating θ if you know **a** and **b**.

Two non-zero vectors are perpendicular (orthogonal) if and only if their dot product is zero, i.e.

$$\begin{aligned}\mathbf{a}.\mathbf{b} = 0 \quad &\Rightarrow \quad |\mathbf{a}||\mathbf{b}|\cos\theta = 0\\ &\Rightarrow \quad \cos\theta = 0\\ &\Rightarrow \quad \theta = \frac{\pi}{2} \quad (90^\circ)\end{aligned}$$

Now consider **i**, **j**, **k**. These are unit vectors, and are mutually perpendicular. These two facts combined show that, e.g.

$$\mathbf{i}\cdot\mathbf{i} = 1, \quad \mathbf{i}\cdot\mathbf{j} = 0, \quad \text{etc.},$$

so if you then let

$$\mathbf{a} = (a_1, a_2, a_3) \qquad (= a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k})$$
$$\mathbf{b} = (b_1, b_2, b_3) \qquad (= b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}),$$

and multiply out $\mathbf{a} \cdot \mathbf{b}$, you obtain

$$\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3.$$

Note:

$$\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}||\mathbf{a}| \cos 0 = |\mathbf{a}|^2$$
$$\text{i.e.} \qquad |\mathbf{a}| = \sqrt{\mathbf{a} \cdot \mathbf{a}}.$$

Let's try this with $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + x\mathbf{k}$. Then:

$$|\mathbf{r}| = \sqrt{\mathbf{r} \cdot \mathbf{r}} = \sqrt{x^2 + y^2 + z^2},$$

which is consistent with the earlier formula for the magnitude of $\mathbf{r}$.

Example 6.4. For

$$\mathbf{a} = 6\mathbf{i} - 3\mathbf{j} + \mathbf{k}$$
$$\mathbf{b} = 4\mathbf{i} + 2\mathbf{j},$$

calculate $\mathbf{a} \cdot \mathbf{b}$ and find the angle between the two vectors.

$$\mathbf{a}.\mathbf{b} = 6 \times 4 + (-3) \times 2 + 1 \times (0) = 18.$$

But recall

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta,$$

and that

$$|\mathbf{a}| = \sqrt{46}, \quad |\mathbf{b}| = 2\sqrt{5},$$

therefore

$$\cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = \frac{18}{2\sqrt{5}\sqrt{46}} = 0.593.$$
$$\therefore \quad \theta = \cos^{-1}(0.593) = 53.6°.$$

Example 6.5. Points A, B and C have coordinates $(3, 2), (4, -3), (7, -5)$ respectively.

i Find $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

ii Find $\overrightarrow{AB}\cdot\overrightarrow{AC}$.

iii Deduce the angle between $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

i

$$\overrightarrow{AB} = (4\mathbf{i} - 3\mathbf{j}) - (3\mathbf{i} + 2\mathbf{j}) = \mathbf{i} - 5\mathbf{j},$$
$$\overrightarrow{AC} = (7\mathbf{i} - 5\mathbf{j}) - (3\mathbf{i} + 2\mathbf{j}) = 4\mathbf{i} - 7\mathbf{j}.$$

ii Now for the dot product:

$$\overrightarrow{AB} \cdot \overrightarrow{AC} = 4 \times 1 + (-5) \times (-7) = 4 + 35 = 39.$$

iii To calculate the angle, note that

$$|\overrightarrow{AB}| = \sqrt{1^2 + (-5)^2} = \sqrt{26},$$
$$|\overrightarrow{AC}| = \sqrt{4^2 + (-7)^2} = \sqrt{65}.$$

Then

$$\cos\theta = \frac{\overrightarrow{AB} \cdot \overrightarrow{AC}}{|\overrightarrow{AB}||\overrightarrow{AC}|} = \frac{39}{\sqrt{26}\sqrt{65}} = 0.949 \quad (3\,\text{d.p.}),$$

which gives $\theta = 18°$.

So far, we have seen one way to multiply two vectors together. However, that first way, the dot product, spits out a number. It would be nice if there was a way to multiply two vectors together such that the result is another vector (Guess what? There is one!)

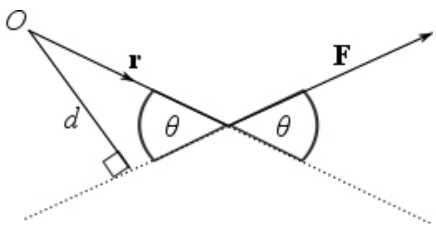
6.3 The Cross Product

Take any two vectors **a** and **b**. Then the *cross product* is denoted as

$$\mathbf{a} \times \mathbf{b}.$$

Before giving the definition, let's consider the motivation behind it using a physics context...

Example 6.6 (Moments)**.** Consider a seesaw. If I apply a force on it at some point away from the pivot, it will turn. Also, if the force is applied farther away from the pivot, the seesaw will turn more easily.



$$\mathbf{r} = \text{Position where the force is exerted}$$
$$\mathbf{F} = \text{The force applied,}$$

then the moment of **F** about a point O is

$$m = |\mathbf{F}|d,$$

where

$$d = |\mathbf{r}|\sin\theta$$

is the perpendicular distance between O and the line of action of $\mathbf{F}$.

$$\therefore \quad m = |\mathbf{r}||\mathbf{F}| \sin\theta.$$

In fact, the moment vector of $\mathbf{F}$ about O, i.e. $\mathbf{m}$, is

$$\mathbf{m} = \mathbf{r} \times \mathbf{F},$$

which is perpendicular to both $\mathbf{r}$ and $\mathbf{F}$. Moreover, $\mathbf{m}$ points in the same direction as the axis of rotation for the seesaw (here, $\mathbf{m}$ points out of the page).

Now, $m = |\mathbf{m}|$, hence the magnitude of $\mathbf{m}$ is:

$$|\mathbf{m}| = |\mathbf{r}||\mathbf{F}| \sin\theta.$$

Okay, now I can define the vector product:

Definition 6.3. *The* cross product *of two vectors* $\mathbf{a}$ *and* $\mathbf{b}$ *is*

$$\mathbf{a} \times \mathbf{b} = |\mathbf{a}||\mathbf{b}| \sin\theta\, \hat{\mathbf{n}},$$

which is a VECTOR, not a NUMBER. So try not to confuse this with the dot product.

$$\begin{aligned} \textit{Length of } \mathbf{a} \times \mathbf{b}: &\quad |\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin\theta. \\ \textit{Direction of } \mathbf{a} \times \mathbf{b}: &\quad \hat{\mathbf{n}}, \textit{ found using the right hand rule.} \end{aligned}$$

$\hat{\mathbf{n}}$ *is a unit vector perpendicular to* $\mathbf{a}$ *and* $\mathbf{b}$.

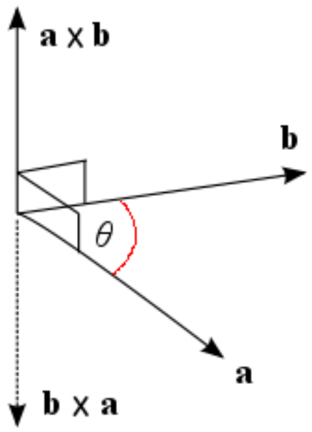


Figure 6.7: The vectors $\mathbf{a}$, $\mathbf{b}$ *and* $\mathbf{a} \times \mathbf{b}$. *If you put your thumb on* $\mathbf{a}$ *and your index finger on* $\mathbf{b}$, *then your middle finger will tell you the direction of* $\mathbf{a} \times \mathbf{b}$.

This definition only works for 3D vectors!

Q: Now, does $\mathbf{a} \times \mathbf{b} = \mathbf{b} \times \mathbf{a}$?

A: NO!

To see this, let $\mathbf{v} = \mathbf{a} \times \mathbf{b}$ and $\mathbf{w} = \mathbf{b} \times \mathbf{a}$. By definition, we will have that $|\mathbf{v}| = |v|$, but what about their directions? Well, the right hand rule shows us that $\mathbf{v} = -\mathbf{w}$. Hence

$$\mathbf{b} \times \mathbf{a} \neq \mathbf{a} \times \mathbf{b}!$$

Suppose we have any two vectors **a** and **b**. If:

$$\begin{aligned} \mathbf{a} &= a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} && = (a_1, a_2, a_3) \\ \mathbf{b} &= b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k} && = (b_1, b_2, b_3), \end{aligned}$$

then the three components of $\mathbf{a} \times \mathbf{b}$ are:

$$\mathbf{a} \times \mathbf{b} = (a_2b_3 - a_3b_2)\mathbf{i} + (a_3b_1 - a_1b_3)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}.$$

This can be conveniently represented using a 3×3 matrix determinant:

$$\mathbf{a} \times \mathbf{b} = \left| \begin{array}{ccc|cc} \mathbf{i} & \mathbf{j} & \mathbf{k} & \mathbf{i} & \mathbf{j} \\ a_1 & a_2 & a_3 & a_1 & a_2 \\ b_1 & b_2 & b_3 & b_1 & b_2. \end{array} \right.$$

A trick to calculate the determinant is to multiply along each of the six diagonal lines. Next, add all the products corresponding to the green diagonals, and then subtract all the products for the red diagonals. In other words,

$$\text{Determinant} = \text{Sum of the green products} - \text{Sum of red products}.$$

Example 6.7. Compute $\mathbf{a} \times \mathbf{b}$, where

$$\begin{aligned} \mathbf{a} &= 4\mathbf{i} - \mathbf{k} \\ \mathbf{b} &= -2\mathbf{i} + \mathbf{j} + 3\mathbf{k} \end{aligned}$$

$$\mathbf{a} \times \mathbf{b} = \left| \begin{array}{ccc|cc} \mathbf{i} & \mathbf{j} & \mathbf{k} & \mathbf{i} & \mathbf{j} \\ 4 & 0 & -1 & 4 & 0 \\ -2 & 1 & 3 & -2 & 1 \end{array} \right.$$

$$\begin{aligned} &= 0\mathbf{i} + 2\mathbf{j} + 4\mathbf{k} - 0\mathbf{k} - (-\mathbf{i}) - 12\mathbf{j} \\ &= \mathbf{i} - 10\mathbf{j} + 4\mathbf{k}. \end{aligned}$$

Example 6.8. Show that $\mathbf{i} \times \mathbf{j} = \mathbf{k}$.

$$\mathbf{i} \times \mathbf{j} = \left| \begin{array}{ccc|cc} \mathbf{i} & \mathbf{j} & \mathbf{k} & \mathbf{i} & \mathbf{j} \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{array} \right.$$

$$\begin{aligned} &= 0\mathbf{i} + 0\mathbf{j} + 1\mathbf{k} - 0\mathbf{i} - 0\mathbf{j} - 0\mathbf{k} \\ &= \mathbf{k}. \end{aligned}$$

Remark 6.1. *A nice interpretation of the length* $|\mathbf{a} \times \mathbf{b}|$ *is that if* θ *is the angle between* **a** *and* **b**, *then this is the area of the parallelogram with sides* **a** *and* **b**, *i.e.*

$$A = \underbrace{|\mathbf{a}|}_{\textit{Base length}} \underbrace{|\mathbf{b}| \sin\theta}_{\textit{Height}}$$

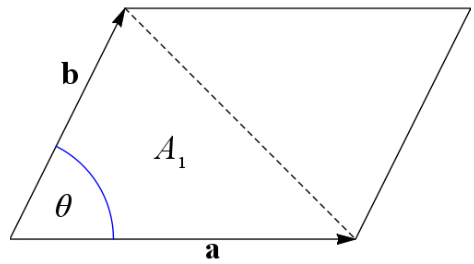


Figure 6.8: A paralellogram, whose sides correspond to vectors **a** *and* **b**. *It can be split into two triangles.*

Proof:

$$A = 2A_1,$$

but

$$\begin{aligned} A_1 &= \frac{1}{2}|\mathbf{a}||\mathbf{b}|\sin\theta, \quad \text{[Anyone recognise this trigonometric formula?]} \\ &= \frac{1}{2}|\mathbf{a}\times\mathbf{b}|, \end{aligned}$$

hence

$$A = |\mathbf{a}\times\mathbf{b}|.$$

□

Example 6.9 (Recycled exam question!)**.** Find the area of a triangle with adjacent sides given by

$$\begin{aligned} \mathbf{a} &= \mathbf{i} + 2\mathbf{j} - \mathbf{k} \\ \mathbf{b} &= \mathbf{j} + \mathbf{k}. \end{aligned}$$

Note that

$$\mathbf{i}\times\mathbf{j} = \left|\begin{array}{ccc|cc} \mathbf{i} & \mathbf{j} & \mathbf{k} & \mathbf{i} & \mathbf{j} \\ 1 & 2 & -1 & 1 & 2 \\ 0 & 1 & 1 & 0 & 1 \end{array}\right.$$

$$\begin{aligned} &= 2\mathbf{i} + 0\mathbf{j} + \mathbf{k} - (-\mathbf{i}) - \mathbf{j} - 0\mathbf{k} \\ &= 3\mathbf{i} - \mathbf{j} + \mathbf{k}. \end{aligned}$$

We want the area of the shaded region A, but

$$\begin{aligned} |\mathbf{a}\times\mathbf{b}| &= 2A \\ \Rightarrow A &= \frac{1}{2}|\mathbf{a}\times\mathbf{b}| \\ &= \frac{1}{2}\sqrt{3^2 + (-1)^2 + 1^2} \\ &= \frac{1}{2}\sqrt{11}. \end{aligned}$$